An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 44 (2002) 1099–1108
www.elsevier.com/locate/camwa

Mixed Equilibrium Problems: Sensitivity Analysis and Algorithmic Aspect

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(Received January 2001; revised and accepted October 2001)

Abstract—The aim of this paper is twofold. First, it is to extend the sensitivity analysis framework, developed recently for variational inequalities, to mixed equilibrium problems. The second is to propose iterative methods for solving this kind of problems. In the process, we establish an equivalence between an extended version of Wiener-Hopf equations and the given problems relying on a generalization of the Yosida approximation notion. Our results generalize results obtained for optimization, variational inequalities, complementarity problems, and problems of Nash equilibria.

Keywords—Equilibrium problems, Sensitivity, Wiener-Hopf equations, Proximal methods.

1. INTRODUCTION AND PRELIMINARIES

Equilibrium problems theory has emerged as an interesting and fascinating branch of applicable mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, operation research in a general and unified way. There is a substantial number of papers on existence results for solving equilibrium problems based on different relaxed monotonicity notions and various compactness assumptions. But up to now no sensitivity analysis and only few iterative methods to solve such problems have been done. Inspired and motivated by the recent research developed for variational inequalities, we consider a class of mixed equilibrium problems which includes variational inequalities as well as complementarity problems, convex optimization, saddle point-problems, problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special cases. Using Wiener Hopf equations technique and adapting ideas of Dafermos [1] and Noor [2], we develop, in the first part of this paper, a sensitivity analysis relying on a fixed point formulation of the given problem. This formulation is obtained thanks to a generalization of the Yosida approximation notion introduced in [3] and allows us to derive, in the second part of this paper, iterative methods for such problems.

Let X be a real Hilbert space and $\|.\|$ the norm generated by the scalar product $\langle .,. \rangle$. Consider the problem of finding

$$\bar{x} \in K; \quad F(g(\bar{x}), y) + \langle T\bar{x}, y - \bar{x} \rangle \geq 0, \qquad \forall y \in K, \tag{MEP}$$

0898-1221/02/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX
PII: S0898-1221(02)00218-3

where K is a nonempty, convex, and closed set of X, T, $g : K \to K$ are two nonlinear operators and $F : K \times K \to \mathbb{R}$ is a given bifunction satisfying $F(x,x) = 0$ for all $x \in K$.

This problem has potential and useful applications in nonlinear analysis and mathematical economics. For example, if we set $F(x,y) = \varphi(y) - \varphi(x)$, for all $x, y \in K$, $\varphi : K \to \mathbb{R}$ a real-valued function, $g = I$ and $T = 0$, then (MEP) reduces to the following minimization problem subject to implicit constraints:

$$\text{find } \bar{x} \in K \text{ such that } \varphi(\bar{x}) \leq \varphi(y), \qquad \forall y \in K. \tag{CO}$$

The basic case of variational inclusions corresponds to $F(x,y) = \sup_{\zeta \in Bx} \langle \zeta, y - x \rangle$ with $B : K \rightrightarrows K$ a set-valued maximal monotone operator. Actually, the mixed equilibrium problem (MEP) is nothing but

$$\text{find } \bar{x} \in K \text{ such that } 0 \in T(\bar{x}) + B(g(\bar{x})), \qquad \forall y \in K. \tag{VP}$$

Moreover, if $F(x,y) = \varphi(y) - \varphi(x)$, then inclusion (VP) reduces to

$$\text{find } \bar{x} \in K \text{ such that } \varphi(y) - \varphi(x) + \langle T(\bar{x}), y - \bar{x} \rangle \geq 0, \qquad \forall y \in K. \tag{VI}$$

In particular if $\varphi = 0$ and K is a closed convex cone, then inequalities (VI) can be written as

$$\text{find } \bar{x} \in K; \qquad T(\bar{x}) \in K^* \quad \text{and} \quad \langle T(\bar{x}), \bar{x} \rangle = 0, \tag{CP}$$

where $K^* = \{x \in X : \langle x, y \rangle \geq 0, \ \forall y \in K\}$ is the polar cone to K. The problem of finding such a $\bar{x}$ is an important instance of the well-known complementarity problem of mathematical programming.

Another example corresponds to Nash equilibria in noncooperative games. Let I (the set of players) be a finite index set. For every $i \in I$ let K_i (the strategy set of the i^{th} player) be a given set, f_i (the loss function of the i^{th} player, depending on the strategies of all players): $K \to \mathbb{R}$ a given function with $K := \prod_{i \in I} K_i$. For $x = (x_i)_{i \in I} \in K$, we define $x^i := (x_j)_{j \in I,\, j \neq i}$. The point $\bar{x} = (\bar{x}_i)_{i \in I} \in K$ is called a Nash equilibrium if and only if for all $i \in I$ the following inequalities hold true:

$$f_i(\bar{x}) \leq f_i(\bar{x}^i, y_i), \qquad \text{for all } y_i \in K_i, \tag{NE}$$

(i.e., no player can reduce his loss by varying his strategy alone). Let $g = I$ and $T = 0$ and define $F : K \times K \to \mathbb{R}$ by

$$F(x,y) = \sum_{i \in I} \left(f_i(x^i, y_i) - f_i(x) \right).$$

Then $\bar{x} \in K$ is a Nash equilibrium if, and only if, $\bar{x}$ solves (MEP). The following definitions and theorem will be needed in the sequel (see for example [4]).

DEFINITION 1. *Let $F : K \times K \to \mathbb{R}$ be a real valued bifunction.*

(i) *F is said to be monotone, if*

$$F(x,y) + F(y,x) \leq 0, \qquad \text{for each } x, y \in K. \tag{1}$$

(ii) *F is said to be strictly monotone if*

$$F(x,y) + F(y,x) < 0, \qquad \text{for each } x, y \in K, \text{ with } x \neq y. \tag{2}$$

(iii) *F is upper-hemicontinuous, if for all x, y, $z \in K$*

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x,y). \tag{3}$$

THEOREM 2. *If the following conditions hold true:*

(i) *F is monotone and upper hemicontinuous,*
(ii) *$F(x,.)$ is convex and lower semicontinuous for each $x \in K$,*
(iii) *there exists a compact subset B of X and there exists $y_0 \in B \cap K$ such that $F(x, y_0) < 0$, for each $x \in K \backslash B$,*

then, the set of solutions to the following problem:

$$\text{find } \bar{x} \in K \text{ such that } F(\bar{x}, y) \geq 0, \qquad \forall\, y \in K, \tag{EP}$$

is nonempty convex and compact.

REMARK 3. If F is strictly monotone, then the solution of (EP) is unique.

2. REGULARIZATION

To begin with, let us recall the extension of the Yosida approximation notion introduced in [3]. Let μ be a positive number. For a given bifunction F the associated Yosida approximation, F_μ, over K and the corresponding regularized operator, A_μ^F, are defined as follows:

$$F_\mu(x,y) = \left\langle \frac{1}{\mu}\left(x - J_\mu^F(x)\right), y - x \right\rangle \qquad \text{and} \qquad A_\mu^F(x) := \frac{1}{\mu}\left(x - J_\mu^F(x)\right), \tag{4}$$

in which $J_\mu^F(x) \in K$ is the unique solution of

$$\mu F\left(J_\mu^F(x), y\right) + \left\langle J_\mu^F(x) - x, y - J_\mu^F(x)\right\rangle \geq 0, \qquad \forall\, y \in K. \tag{5}$$

REMARK 4. The existence and uniqueness of the solution of problem (5) follow by invoking Theorem 2 and Remark 3.

Observe that in the case where $F(x,y) = \sup_{\zeta \in Bx} \langle \zeta, y - x\rangle$ and $K = X$, B being a maximal monotone operator, it directly yields

$$J_\mu^F(x) = (I + \mu B)^{-1} x \qquad \text{and} \qquad A_\mu^F(x) = B_\mu(x),$$

where $B_\mu := (1/\mu)\left(I - (I + \mu B)^{-1}\right)$ is the Yosida approximation of B, and we recover the classical concepts.

Before giving some properties of the Yosida approximation, let us recall that a mapping M is c-firmly nonexpansive (c being a positive constant) if for all $x,\ y \in X$

$$|M(x) - M(y)|^2 \leq |x - y|^2 - c|(I - M)x - (I - M)y|^2.$$

LEMMA 5. *Assume that conditions of Theorem 2 are fulfilled, then the operator J_μ^F is cocoercive with modulus 1, that is,*

$$\left\langle J_\mu^F(x) - J_\mu^F(y), x - y\right\rangle \geq \left|J_\mu^F(x) - J_\mu^F(y)\right|^2, \qquad \forall\, x, y \in X, \tag{6}$$

J_μ^F is 1-firmly nonexpansive, namely,

$$\left|J_\mu^F(x) - J_\mu^F(y)\right|^2 \leq |x - y|^2 - \left|\left(I - J_\mu^F\right)x - \left(I - J_\mu^F\right)y\right|^2, \tag{7}$$

and A_μ^F is cocoercive with modulus μ, namely,

$$\left\langle A_\mu^F(x) - A_\mu^F(y), x - y\right\rangle \geq \mu\left|A_\mu^F(x) - A_\mu^F(y)\right|^2, \qquad \forall\, x, y \in X. \tag{8}$$

PROOF. From relation (5), we can write

$$\mu F\left(J_\mu^F(x), J_\mu^F(y)\right) + \left\langle J_\mu^F(x) - x, J_\mu^F(y) - J_\mu^F(x)\right\rangle \geq 0, \qquad \forall x, y \in K,$$

and

$$\mu F\left(J_\mu^F(y), J_\mu^F(x)\right) + \left\langle J_\mu^F(y) - y, J_\mu^F(x) - J_\mu^F(y)\right\rangle \geq 0, \qquad \forall x, y \in K.$$

By adding the last two inequalities and using the monotonicity of F, we obtain the desired result. Equation (7) follows from (6), indeed we have successively

$$\begin{aligned}\left|\left(I - J_\mu^F\right) x - \left(I - J_\mu^F\right) y\right|^2 &= |x-y|^2 - 2\left\langle J_\mu^F(x) - J_\mu^F(y), x - y\right\rangle + \left|J_\mu^F(x) - J_\mu^F(y)\right|^2 \\ &\leq |x-y|^2 - \left|J_\mu^F(x) - J_\mu^F(y)\right|^2.\end{aligned}$$

Now combining (6) with

$$x = J_\mu^F(x) + \lambda A_\mu^F(x) \qquad \text{and} \qquad y = J_\mu^F(y) + \lambda A_\mu^F(y),$$

we obtain

$$\left\langle A_\mu^F(x) - A_\mu^F(y), J_\mu^F(x) - J_\mu^F(y)\right\rangle \geq 0. \tag{9}$$

On the other hand

$$\begin{aligned}\left\langle A_\mu^F(x) - A_\mu^F(y), x - y\right\rangle = &\left\langle A_\mu^F(x) - A_\mu^F(y), x - J_\mu^F(x) - \left(y - J_\mu^F(y)\right)\right\rangle \\ &+ \left\langle A_\mu^F(x) - A_\mu^F(y), J_\mu^F(x) - J_\mu^F(y)\right\rangle.\end{aligned}$$

The announced result follows by noticing that

$$\left\langle A_\mu^F(x) - A_\mu^F(y), x - J_\mu^F(x) - \left(y - J_\mu^F(y)\right)\right\rangle = \mu\left|A_\mu^F(x) - A_\mu^F(y)\right|^2.$$

Now, in relation to the mixed problem (MEP), we consider the following equation:

$$\text{find } z \in X \text{ such that } T(x) + A_\mu^F(z) = 0 \text{ and } g(x) = J_\mu^F z. \tag{GWH}$$

REMARK 6. It is easy to see that the mixed equilibrium problem has a solution x if and only if the generalized Wiener Hopf equation has a solution z, where

$$g(x) = J_\mu^F z \qquad \text{and} \qquad z = g(x) - \mu T(x). \tag{10}$$

This follows immediately from the next fixed-point formulation:

$$x \text{ solves (MEP) if and only if } g(x) = J_\mu^F(g(x) - \mu T(x)).$$

In the sequel, we assume that the bifunction F satisfies conditions of Theorem 2.

3. SENSITIVITY ANALYSIS

Now, we consider the parametric versions of problems (MEP) and (GWH). To formulate the problems, let Λ be an open subset of a Hilbert space Y in which λ takes values and $|.|$ the norm generated by its scalar product, then the parametric version of (MEP) is given by

$$\text{find } \bar{x}_\lambda \in X \text{ such that } 0 \in F\left(g\left(\bar{x}_\lambda, \lambda\right), y, \lambda\right) + \left\langle T\left(\bar{x}_\lambda, \lambda\right), y - \bar{x}\right\rangle \geq 0,$$

where $T(.,\lambda) : K \times \Lambda \to K$, $F(.,\lambda) : (K \times K) \times \Lambda \rightrightarrows K$ are given bifunctions.

The associated Wiener-Hopf equation is

$$\text{find } z_\lambda \in X; \qquad T(x_\lambda, \lambda) + \left(A_\mu^F(.,\lambda)\right)_\mu z_\lambda = 0 \quad \text{and} \quad g(x_\lambda, \lambda) = J_\mu^{F(.,\lambda)} z_\lambda.$$

We suppose that for some $\bar{\lambda} \in \Lambda$, problem (11) has a unique solution $\bar{x}$ which is equivalent, by Remark 6, to assuming that (12) has a unique solution $\bar{z}$. In what follows, we are interested in knowing if (11) (respectively, (12)) has a solution, denoted x_λ (respectively, z_λ), close to $\bar{x}$ (respectively, $\bar{z}$) when λ is close to $\bar{\lambda}$, and how the function $x(\lambda) := x_\lambda$ (respectively, $z(\lambda) := z_\lambda$) behaves. In other words, we want to investigate the sensitivity of the solutions $\bar{x}$, $\bar{z}$ with respect to the change of the parameter λ.

Now let ϑ be a closed convex neighbourhood of $\bar{z}$. We will use the alternative fixed-point formulation given in Remark 6 to study the sensitivity of problems (11) and (12). More precisely, we want to investigate those conditions under which, for each z_λ near $\bar{z}$ (respectively, x_λ near $\bar{x}$) the function $z_\lambda := z(\lambda)$ (respectively, $x_\lambda := x(\lambda)$) is continuous or Lipschitz continuous.

DEFINITION 7. *Let T be an operator defined on $\vartheta\times\Lambda$. Then for all $x,\ y \in \vartheta$, the operator is said to be*

(i) *locally strongly monotone if there exists a constant $\alpha > 0$ such that*

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geq \alpha\|x-y\|^2, \tag{13}$$

(ii) *locally Lipschitz continuous if there exists a constant $\beta > 0$ such that*

$$\|T(x,\lambda) - T(y,\lambda)\| \leq \beta\|x-y\|.$$

It is clear that $\alpha \leq \beta$.

We consider the case, when the solutions of the parametric Weiner-Hopf equation (12) lie in the interior of ϑ. Following the ideas of Dafermos [1], we consider the map

$$\begin{aligned} G(z,\lambda) &= J_\mu^{F_{|\vartheta}(.,\lambda)}(z_\lambda) - \mu T((x_\lambda,\lambda)) \\ &= g(x_\lambda,\lambda) - \mu T(x_\lambda,\lambda), \end{aligned} \tag{15}$$

where $g(x_\lambda,\lambda) = J_\mu^{F_{|\vartheta}(.,\lambda)} z_\lambda$ and $F_{|\vartheta} : (K\cap\vartheta \times K\cap\vartheta)\times\Lambda \rightrightarrows X$ with $F_{|\vartheta} = F$.

We have to show that the map $z \to G(z,\lambda)$ has a fixed point, which is also a solution of (12). First of all, we prove that the map is a contraction with respect to z, uniformly in $\lambda \in \Lambda$ by using Assumptions (i) and (ii) on the operators $T(.,\lambda)$ and $g(.,\lambda)$ defined on $\vartheta\times\Lambda$.

LEMMA 8. *Let the operator $T(.,\lambda)$ be locally strongly monotone with constant α, locally Lipschitz continuous with constant β and $g(.,\lambda)$ be locally strongly monotone with constant δ, locally Lipschitz continuous with constant σ, if*

$$1-k > 0, \qquad \alpha > 2\beta\sqrt{k(1-k)}, \qquad \text{and} \qquad \left|\mu - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - 4k(1-k)\beta^2}}{\beta^2},$$

then for all $z_1, z_2 \in \vartheta$ and $\lambda \in \Lambda$, we have

$$\|G(z_1,\lambda) - G(z_2,\lambda)\| \leq \theta\|z_1 - z_2\|, \tag{17}$$

$$\text{where } k := \sqrt{1-2\delta+\sigma^2} \text{ and } \theta := \frac{k + \sqrt{1-2\mu\alpha+\mu^2\beta^2}}{1-k}. \tag{18}$$

PROOF. For all $z_1, z_2 \in \vartheta$, $\lambda \in \Lambda$, by (15) and the triangular inequality, we get

$$\begin{aligned} \|G(z_1,\lambda) - G(z_2,\lambda)\| \leq{} & \|x_1 - x_2 - (g(x_1,\lambda) - g(x_2,\lambda))\| \\ & + \|x_1 - x_2 - \mu(T(x_1,\lambda) - T(x_2,\lambda))\|. \end{aligned} \tag{19}$$

Set $E = \|x_1 - x_2 - (g(x_1,\lambda) - g(x_2,\lambda))\|^2$, since $g(.,\lambda)$ is strongly monotone and Lipschitz continuous, it follows that

$$\begin{aligned} E ={} & \|x_1 - x_2\|^2 - 2\mu\langle g(x_1,\lambda) - g(x_2,\lambda), x_1 - x_2\rangle \\ & + \mu^2\|g(x_1,\lambda) - g(x_2,\lambda)\|^2 \leq \left(1 - 2\delta + \sigma^2\right)\|x_1 - x_2\|^2. \end{aligned} \tag{20}$$

Similarly,

$$\|x_1 - x_2 - \mu(T(x_1) - T(x_2))\|^2 \leq \left(1 - 2\mu\beta + \mu^2\gamma^2\right)\|x_1 - x_2\|^2. \tag{21}$$

From (19)–(21), we obtain

$$\|G(z_1,\lambda) - G(z_2,\lambda)\| \leq \left(\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\mu\alpha+\mu^2\beta^2}\right)\|x_1 - x_2\|. \tag{22}$$

According to (12) and using the nonexpansivness of the extended resolvent, we can write

$$\begin{aligned}\|x_1 - x_2\| &\le \left\|x_1 - x_2 - (g(x_1,\lambda) - g(x_2,\lambda)) + J_\mu^{F_{|\vartheta}(.,\lambda)} z_1 - J_\mu^{F_{|\vartheta}(.,\lambda)} z_2\right\| \\ &\le k\|x_1 - x_2\| + \|z_1 - z_2\|.\end{aligned}$$

Thus,

$$\|x_1 - x_2\| \le \frac{1}{1-k}\|z_1 - z_2\|,$$

which combined with (22) yields

$$\|G(z_1,\lambda) - G(z_2,\lambda)\| \le \theta\|z_1 - z_2\|.$$

Since $\theta < 1$ for μ satisfying (16), it follows that the map $z \to G(z,\lambda)$ is a contraction and has a fixed point $z(\lambda)$, the solution of the parametric Wiener-Hopf equations (12).

REMARK 9. Since $\bar{z}$ is a solution of (12) for $\lambda = \bar{\lambda}$, it is then easy to show that $\bar{z}$ is the unique fixed point in ϑ of the map $G(.,\bar{\lambda})$. In other words,

$$\bar{z} = z\left(\bar{\lambda}\right) = G\left(z\left(\bar{\lambda}\right), \bar{\lambda}\right). \tag{23}$$

Using Lemma 8, we prove the continuity of the solution $z(\lambda)$ (respectively, $x(\lambda)$) of (12) (respectively, (11)) which is the main motivation of the next result.

LEMMA 10. *If the operators $T(x,.)$ and $g(.,\lambda)$ are continuous (or Lipschitz continuous), then the function $z(\lambda)$ is continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$. If in addition the map $\lambda \to J_\mu^{F_{|\vartheta}(.,\lambda)}\bar{z}$ is continuous (or Lipschitz continuous), the function $x(\lambda)$ is in turn continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$.*

PROOF. For $\lambda \in \Lambda$ using Lemma 8 and the triangular inequality, we have

$$\begin{aligned}\left\|z(\lambda) - z\left(\bar{\lambda}\right)\right\| &= \left\|G(z(\lambda),\lambda) - G\left(\bar{z},\bar{\lambda}\right)\right\| \\ &\le \|G(z(\lambda),\lambda) - G(\bar{z},\lambda)\| + \left\|G(\bar{z},\lambda) - G\left(\bar{z},\bar{\lambda}\right)\right\| \\ &\le \theta\|z(\lambda) - \bar{z}\| + \left\|G(\bar{z},\lambda) - G\left(\bar{z},\bar{\lambda}\right)\right\|.\end{aligned} \tag{24}$$

On the other hand, from (15)

$$\left\|G(\bar{z},\lambda) - G\left(\bar{z},\bar{\lambda}\right)\right\| = \left\|g(\bar{x},\lambda) - g(\bar{x},\lambda) - \mu\left(T(\bar{x},\lambda) - T\left(\bar{x},\bar{\lambda}\right)\right)\right\|. \tag{25}$$

Combining the last inequality of (24) and relation (25), we obtain

$$\|z(\lambda) - \bar{z}\| \le \frac{1}{1-\theta}\left(\mu\left\|T(\bar{x},\lambda) - T\left(\bar{x},\bar{\lambda}\right)\right\| + \left\|g(\bar{x},\lambda) - g\left(\bar{x},\bar{\lambda}\right)\right\|\right), \tag{26}$$

from which the the first part of the desired result follows.

Now, we have

$$\begin{aligned}\left\|x(\lambda) - x\left(\bar{\lambda}\right)\right\| &\le \|x(\lambda) - \bar{x} - (g(x(\lambda),\lambda) - g(\bar{x},\lambda))\| + \|g(x(\lambda),\lambda) - g(\bar{x},\lambda)\| \\ &\le k\|x(\lambda) - \bar{x}\| + \left\|g(x(\lambda),\lambda) - g\left(\bar{x},\bar{\lambda}\right)\right\| + \left\|g\left(\bar{x},\bar{\lambda}\right) - g(\bar{x},\lambda)\right\|.\end{aligned}$$

Since $g(x(\lambda)) = J_\mu^{F_{|\vartheta}(.,\lambda)}z(\lambda)$ and $g(\bar{x}) = g(x(\bar{\lambda})) = J_\mu^{F_{|\vartheta}(.,\bar{\lambda})}\bar{z}$, we can write

$$\left\|x(\lambda) - x\left(\bar{\lambda}\right)\right\| \le \frac{1}{1-k}\left(\|z(\lambda) - \bar{z}\| + \left\|J_\mu^{F_{|\vartheta}(.,\lambda)}\bar{z} - J_\mu^{F_{|\vartheta}(.,\bar{\lambda})}\bar{z}\right\| + \left\|g(\bar{x},\lambda) - g\left(\bar{x},\bar{\lambda}\right)\right\|\right).$$

This combined with (26), yields

$$\begin{aligned}\left\|x(\lambda) - x\left(\bar{\lambda}\right)\right\| \le \frac{1}{1-k}&\left(\frac{\mu}{1-\theta}\left\|A(\bar{x},\lambda) - T\left(\bar{x},\bar{\lambda}\right)\right\| + \frac{2-\theta}{1-\theta}\left\|g(\bar{x},\lambda) - g\left(\bar{x},\bar{\lambda}\right)\right\|\right) \\ &+ \frac{1}{1-k}\left\|J_\mu^{F_{|\vartheta}(.,\lambda)}\bar{z} - J_\mu^{F_{|\vartheta}(.,\bar{\lambda})}\bar{z}\right\|,\end{aligned}$$

from which we obtain the required result.

LEMMA 11. *If the assumptions of Lemma 10 above hold true, then there exists a neighbourhood $\aleph \subset \Lambda$ of $\bar{\lambda}$ such that for all $\lambda \in \aleph$, $z(\lambda)$ (respectively, $x(\lambda)$) is the unique solution of (12) (respectively, (11)) in the interior of ϑ.*

PROOF. Similar to Lemma 2.5 in [1].

We now state and prove the main result of the first part of this paper.

THEOREM 12. *Let $\bar{x}$ be the solution of the parametric mixed equilibrium problem (11) and $\bar{z}$ the solution of the parametric Wiener-Hopf equations (12) for $\lambda = \bar{\lambda}$. Let $T(.,\lambda)$, $g(.,\lambda)$ be locally strongly monotone and locally Lipschitz continuous operators on ϑ. If the operators $T(\bar{x},.)$, $g(\bar{x},.)$ are continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$, then there exists a neighbourhood $\aleph \subset \Lambda$ of $\bar{\lambda}$ such that for $\lambda \in \aleph$, (12) has a unique solution $z(\lambda)$ in the interior of ϑ, $z(\bar{\lambda}) = \bar{z}$, and $z(\lambda)$ is continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$.*

If in addition the map $\lambda \to J_{\mu}^{F_{|\vartheta}(.,\lambda)}\bar{z}$ is continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$, then for $\lambda \in \aleph$ the parametric problem (11) has a unique solution $x(\lambda)$ in the interior of ϑ, $x(\bar{\lambda}) = \bar{x}$, and $x(\lambda)$ is continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$.

PROOF. Its proof follows from Lemmas 8, 10 and 11, and Remark 9.

REMARK 13. Among others, in the special case where $G(x,y,\lambda) := \delta_{K_\lambda}(y) - \delta_{K_\lambda}(x)$, δ_{K_λ} the indicator function of a closed convex set K_λ and $g(.,\lambda) = I$, (11) reduces to

$$\text{find } x_\lambda \in X; \quad \langle T(x_\lambda,\lambda), y - x_\lambda\rangle \geq 0, \qquad \text{for all } y \in K_\lambda,$$

and we recover the main result of Noor [5].

4. ALGORITHMIC ASPECT

To begin with, let us observe that the projection and resolvent methods cannot be used to suggest iterative methods for equilibrium problems of type (MEP) due to the presence of the term F.

4.1. The Basic Algorithm

To introduce the basic algorithm, we use the alternative fixed-point formulation of Remark 6. More precisely, we apply the successive approximation method to solving $x = H(x)$, where

$$H(x) = x - g(x) + J_{\mu}^{F}(g(x) - \mu T(x)).$$

The resulting procedure follows.

ALGORITHM. *Given the iterate x_k compute the point x_{k+1} by*

$$x_{k+1} = x_k - g(x_k) + J_{\mu}^{F}(g(x_k) - \mu T(x_k)), \tag{27}$$

where $\mu > 0$ is a constant.

The next result gives convergence property of the above algorithm.

THEOREM 14. *Let the operator T be strongly monotone with constant α, Lipschitz continuous with constant β and g be strongly monotone with constant δ, Lipschitz continuous with constant σ. If the parameters satisfy condition (16), then x_{k+1} is well defined and the sequence $\{x_k\}$ strongly converges to a solution of (MEP).*

PROOF. Observe that any solution x of problem (MEP) is a fixed point of H. Then, thanks to the triangular inequality and to the fact that the mapping J_{μ}^{F} is nonexpansive, we have

$$\|x_{k+1} - x\| \leq 2\|x_k - x - (g(x_k) - g(x))\| + \|x_k - x - \mu(T(x_k) - T(x))\|. \tag{28}$$

In the proof of Lemma 8, we have shown that

$$\|x_k - x - (g(x_k) - g(x))\| \le \sqrt{1 - 2\delta + \sigma^2}\, \|x_k - x\|$$

and

$$\|x_1 - x_2 - \mu(T(x_1) - T(x_2))\|^2 \le \sqrt{1 - 2\mu\alpha + \mu^2\beta^2}\, \|x_k - x\|.$$

Hence,

$$\|x_{k+1} - x\| \le \gamma \|x_k - x\|, \tag{29}$$

where $\gamma = (2k + \sqrt{1 - 2\mu\alpha + \mu^2\beta^2})$ and $k = \sqrt{1 - 2\delta + \sigma^2}$.

Since $\gamma < 1$ for μ satisfying (16), we deduce from relation (29) that the sequence $\{x_k\}$ strongly converges to x.

For $g = I$ we can give a weaker convergence result. This is the motivation of the next theorem.

THEOREM 15. *Let T be a cocoercive operator with constant γ. If $\mu < 2\gamma$, then the sequence $\{x_k\}$ weakly converges to a solution of (MEP).*

PROOF. Set $H_\mu(x) := J_\mu^F(x - \mu T x)$. First, observe that the mapping $I - \mu T$ is $(2\gamma/\mu - 1)$-firmly nonexpansive. Indeed, since T is cocoercive, we have successively

$$\begin{aligned} |(I - \mu T)(x) - (I - \mu T)(y)|^2 &\le |x - y|^2 +^2 |\mu T x - \mu T y|^2 - 2\gamma\mu |Tx - Ty|^2 \\ &\le |x - y|^2 - \left(\frac{2\gamma}{\mu} - 1\right) |\mu T x - \mu T y|^2. \end{aligned}$$

Then, as the composition of the 1-firmly nonexpansive mapping J_μ^F with the $(2\gamma/\mu - 1)$-firmly nonexpansive mapping $I - \mu T$, H_μ is c-firmly nonexpansive with $c := \min\{1, 2\gamma/\mu - 1\}/2$, thanks to [6, Lemma 3.1]. Furthermore, for all x^* solution of (MEP), which is a fixed point of H_μ, we have

$$|x_{k+1} - x^*|^2 \le |x_k - x^*|^2 - c\, |x_k - x_{k+1}|^2,$$

thus, for all $k \in \mathbb{N}$,

$$|x_{k+1} - x^*| \le |x_k - x^*|.$$

The sequence of iterates $\{x_k\}$ is therefore, bounded, $\lim_{k \to +\infty} |x_k - x^*|$ exists and is finite and $\{x_k\}$ is asymptotically regular, that is $\lim_{k \to +\infty} |x_k - x_{k+1}| = 0$. On the other hand, from (5) and (27) and monotonicity of F, we derive

$$\left\langle \frac{1}{\mu}(x_{k+1} - x_k) + T x_k, y - x_{k+1} \right\rangle \ge F(y, x_{k+1}), \qquad \forall y \in K.$$

Passing to the limit (on a subsequence) in the last inequality and taking into account the lower semicontinuity of F, we obtain

$$\langle T\tilde{x}, y - \tilde{x} \rangle \ge F(y, \tilde{x}), \qquad \forall y \in K,$$

where $\tilde{x}$ is any weak limit point.

Now, let $x_t = ty + (1 - t)\tilde{x}, 0 < t \le 1$, from the properties of F follows then for all t

$$\begin{aligned} 0 = F(x_t, x_t) &\le tF(x_t, y) + (1 - t) F(x_t, \tilde{x}) \\ &\le tF(x_t, y) + (1 - t) \langle T\tilde{x}, x_t - \tilde{x} \rangle \\ &= tF(x_t, y) + (1 - t) t \langle T\tilde{x}, y - \tilde{x} \rangle. \end{aligned}$$

Dividing by t and letting $t \downarrow 0$ we get $x_t \to \tilde{x}$ which together with the upper hemicontinuity of F yields

$$F(\tilde{x}, x) + \langle T\tilde{x}, y - \tilde{x} \rangle \ge 0, \qquad \forall x \in K,$$

that is any weak limit point $\tilde{x}$ is solution to the problem (MEP). The uniqueness of such limit point is standard (see for example [7, Theorem 1]).

REMARK 16.

(i) Observe that when the operator is a gradient, then the cocoercivity property is equivalent to simple monotonicity plus Lipschitz continuity. But, this equivalence is not true in general, indeed the rotation by $\pi/2$, is not cocoercive although it is monotone and Lipschitz.

(ii) It is easy to check that when an operator is strongly monotone and Lipschitz, then it is cocoercive. So we may say that for Lipschitz operators, the cocoercivity is weaker than the strong monotonicity.

4.2. Special Cases

4.2.1. Convex optimization

Let φ be a convex, proper, and lower semicontinuous function, then if we take $F(x,y) = \varphi(y) - \varphi(x)$, $g = I$, $T = 0$, and $K = X$, our method reduces to

$$x_{k+1} = J_\mu^\varphi(x_k) = (I + \mu\partial\varphi)^{-1}(x_k),$$

that is

$$\frac{1}{\mu}(x_k - x_{k+1}) \in \partial\varphi(x_{k+1}),$$

where $\partial\varphi$ stands for the convex subdifferential of φ.

From which we infer

$$x_{k+1} = \arg\min\left\{\varphi(x) + \frac{1}{2\varepsilon}|x - x_k|^2\right\}.$$

So we recover Martinet's regularization method proposed in [8] and its convergence results as a special case.

4.2.2. Monotone inclusions

First, let us notice that by taking $F(x,y) = \sup_{\xi\in Bx}\langle \xi, y-x\rangle$ for all $y,\ x \in K$, where $B : K \rightrightarrows X$ is a maximal monotone operator, (MEP) is nothing but the problem of finding a zero of the operator $B+T$. On the other hand, F is maximal monotone according to Blum's-Oetlli definition, namely, for every $(\zeta, x) \in X \times K$

$$F(y,x) \le \langle -\zeta, y - x\rangle, \qquad \forall\, y \in K \Rightarrow 0 \le F(x,y) + \langle -\zeta, y - x\rangle, \quad \forall\, y \in K.$$

It should be noticed that a monotone function which is convex in the second argument and upper hemicontinuous in the first one is maximal monotone.

Moreover, by taking $K = X$, $F(x,y) = \sup_{\xi\in Bx}\langle \xi, y - x\rangle$, and $g = I$, our method reduces to the following scheme (see [9]):

$$x_{k+1} = (I + \mu B)^{-1}(x_k - \mu T(x_k)), \tag{31}$$

and we recover the classical result (see also [10]).

REMARK 17.

(i) We would like to mention that we could consider a perturbed version of the basic algorithm introduced above by replacing the function F by a sequence of function F_k which approximates F in the following sense:

$$\lim_{k\to\infty} J_\mu^{F_k}(x) = J_\mu^F(x), \qquad \text{for all } x \in X. \tag{32}$$

Under condition (32) the convergence result still holds true. Furthermore, we observe that where $F_k(x,y) = \sup_{\xi \in B_k x} \langle \xi, y-x \rangle$ and $F(x,y) = \sup_{\xi \in Bx} \langle \xi, y-x \rangle$, B_k, B being two maximal monotone operators, the above condition is equivalent to the graph convergence of B_k to B, that is

$$\forall x, y \in X \text{ with } y \in Bx,\ \exists\, x_k, y_k \in X \text{ such that } y_k \in Bx_k,\ x_k \to x \text{ and } y_k \to y,$$

(see [11]).

(ii) It is worth mentioning that the analysis developed here can be applied to set-valued mixed equilibrium problems, the corresponding fixed-point formulation, which is very important from numerical and approximation point of views, will enable us to suggest a number of iterative algorithms.

(iii) The Yosida approximation notion considered in this paper is the right concept, it allows us to derive the new flexible alternative formulation of the given problem and it reduces to the classical notion when the function is associated with a maximal monotone operator.

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